S and **Q** Matrices Reloaded

Applications to Open, Inhomogeneous and Complex Cavities

G. Painchaud-April, **J. Dumont**,
D. Gagnon and L. J. Dubé

Département de physique, génie physique et optique
Université Laval, Québec, Canada

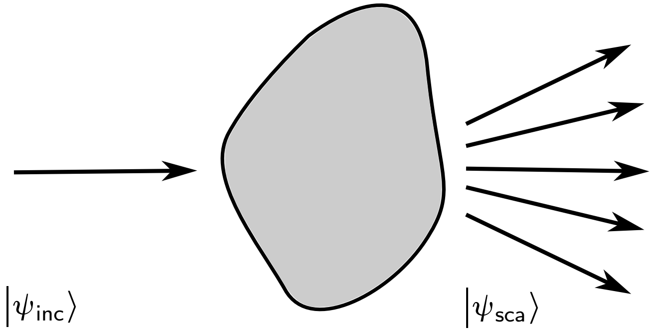
ICTON 2013
Cartagena, Spain

Outline

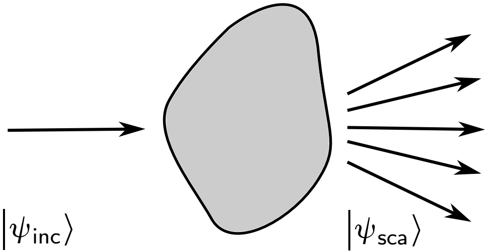
Section Outline

Section Outline

What is Resonance?



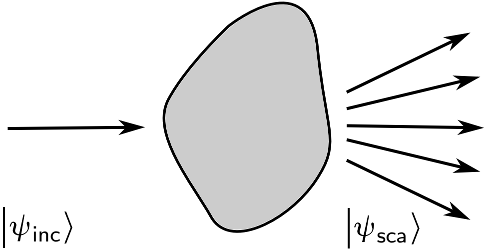
What is Resonance?



- Outside the dielectric, we solve Helmholtz' equation:

$$\left[\nabla^2 + n^2k^2\right]|\psi\rangle = 0.$$

What is Resonance?

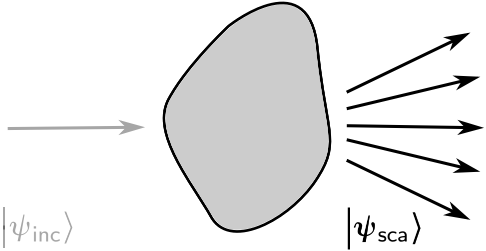


- Outside the dielectric, we solve Helmholtz' equation:

$$\left[\nabla^2 + n^2k^2\right]|\psi\rangle = 0.$$

- $|\psi_{\text{sca}}\rangle = \mathscr{S}\,|\psi_{\text{inc}}\rangle$

What is Resonance?



- Outside the dielectric, we solve Helmholtz' equation:

$$\left[\nabla^2 + n^2k^2\right]|\psi\rangle = 0.$$

- $|\psi_{\mathrm{sca}}\rangle = \mathscr{S}\,|\psi_{\mathrm{inc}}\rangle$
- Poles of $|\det \boldsymbol{S}(k)|$ with $k \in \mathbb{C}$ ($\boldsymbol{S}(k)$ = representation of $\mathscr{S}$).

What Is Resonance?

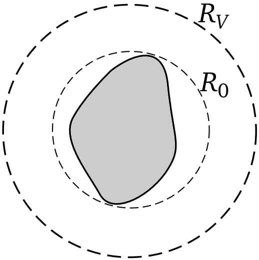
Can we find a set of real energy levels for an open dielectric cavity?

Section Outline

Delay due to the Cavity

Average E.M. Energy

$$\mathscr{E}^V = \frac{1}{2} \iiint_{R_V} \left[\epsilon \boldsymbol{E}^* \cdot \boldsymbol{E} + \mu \boldsymbol{H}^* \cdot \boldsymbol{H} \right] d^3\boldsymbol{r}$$



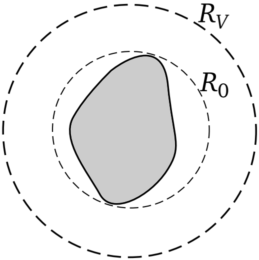
Delay due to the Cavity

Average E.M. Energy

$$\mathscr{E}^V = \frac{1}{2}\iiint_{R_V} \left[\epsilon \boldsymbol{E}^* \cdot \boldsymbol{E} + \mu \boldsymbol{H}^* \cdot \boldsymbol{H}\right] d^3\boldsymbol{r}$$

- Outside the cavity ($r > R_0$), the modes are

$$\psi_m = H_m^{(-)}(n_o kr)e^{im\theta} + \sum_\ell S_{m\ell} H_\ell^{(+)}(n_o kr)e^{i\ell\theta}$$



Delay due to the Cavity

Average E.M. Energy

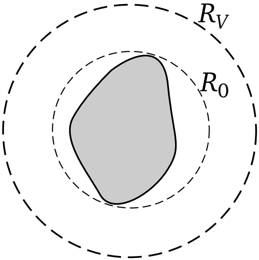
$$\mathcal{E}^V = \frac{1}{2}\iiint_{R_V} \left[\epsilon \boldsymbol{E}^* \cdot \boldsymbol{E} + \mu \boldsymbol{H}^* \cdot \boldsymbol{H}\right] d^3\boldsymbol{r}$$

- Outside the cavity ($r > R_0$), the modes are

$$\psi_m = H_m^{(-)}(n_o kr)e^{im\theta} + \sum_\ell S_{m\ell} H_\ell^{(+)}(n_o kr)e^{i\ell\theta}$$

- Coupling between modes:

$$\mathcal{E}^V_{mm'} = \frac{1}{2}\iiint_{R_V} \Big[\epsilon \boldsymbol{E}^*_m \cdot \boldsymbol{E}_{m'} + \mu \boldsymbol{H}^*_m \cdot \boldsymbol{H}_{m'}\Big] d^3\boldsymbol{r}$$



Delay due to the Cavity

$$\mathcal{E}^{\infty}_{mm'} = \lim_{R_V \to \infty} \frac{4 n_0 R_V w}{k} \delta_{mm'} + \frac{2w}{k} \left(-i \sum_{\ell} S^*_{\ell m} \frac{\partial S_{\ell m'}}{\partial k} \right)$$

Delay due to the Cavity

$$\mathcal{E}^{\infty}_{mm'} = \lim_{R_V \to \infty} \frac{4n_0 R_V w}{k}\delta_{mm'} + \frac{2w}{k}\left(-i\sum_{\ell} S^{*}_{\ell m}\frac{\partial S_{\ell m'}}{\partial k}\right)$$

$\mathcal{E}^{0}_{mm'}$: diverging free space energy of the beam.

Delay due to the Cavity

$$\mathcal{E}^{\infty}_{mm'} = \lim_{R_V \to \infty} \frac{4 n_0 R_V w}{k} \delta_{mm'} + \frac{2w}{k} \left(-i \sum_{\ell} S^{*}_{\ell m} \frac{\partial S_{\ell m'}}{\partial k} \right)$$

$\mathcal{E}^{0}_{mm'}$: diverging free space energy of the beam.

$\mathbf{Q} = -i\mathbf{S}^{\dagger} \frac{\partial \mathbf{S}}{\partial k}$: complex coupling between angular momentum channels → excess energy due to the cavity → delay times.

Section Outline

Resonances $\longleftrightarrow$ Time Delay

Characteristic Modes of the Cavity

- The field is given by

$$\psi^p = \sum_m \left[A_m^p H_m^{(-)}(nkr) + B_m^p H_m^{(+)}(nkr) \right] e^{im\theta}.$$

and

$$\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}.$$

Resonances $\longleftrightarrow$ Time Delay

Characteristic Modes of the Cavity

- The field is given by

$$\psi^p = \sum_m \left[A_m^p H_m^{(-)}(nkr) + B_m^p H_m^{(+)}(nkr) \right] e^{im\theta}.$$

and

$$\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}.$$

- Setting up the eigenvalue problem:

$$\mathbf{Q}\boldsymbol{A}^p = \tau_p \boldsymbol{A}^p.$$

Resonances $\longleftrightarrow$ Time Delay

Characteristic Modes of the Cavity

- The field is given by

$$\psi^p = \sum_m \left[A_m^p H_m^{(-)}(nkr) + B_m^p H_m^{(+)}(nkr) \right] e^{im\theta}.$$

  and

$$\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}.$$

- Setting up the eigenvalue problem:

$$\mathbf{Q}\boldsymbol{A}^p = \tau_p \boldsymbol{A}^p.$$

- τ_p are the delay times and represent the time the energy of the mode described by $\boldsymbol{A}^p$ spends inside the cavity.

Resonances $\longleftrightarrow$ Time Delay

Characteristic Modes of the Cavity

- The field is given by

$$\psi^p = \sum_m \left[A_m^p H_m^{(-)}(nkr) + B_m^p H_m^{(+)}(nkr) \right] e^{im\theta}.$$

and

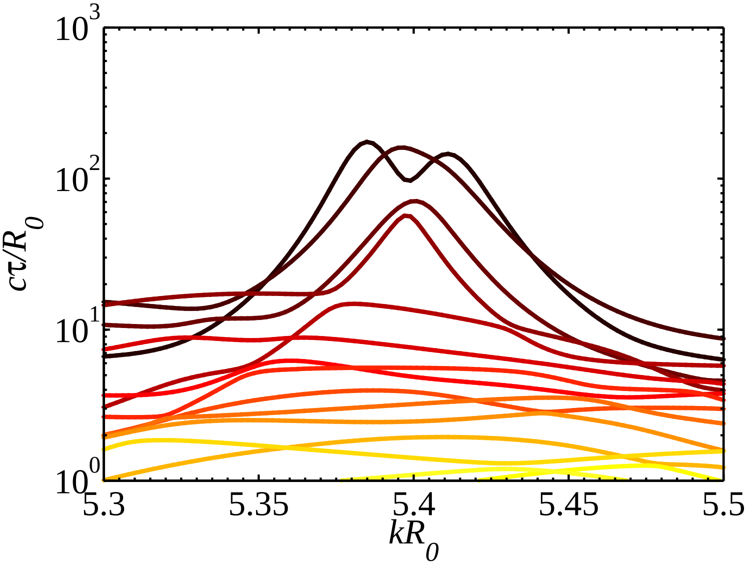
$$\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}.$$

- Setting up the eigenvalue problem:

$$\mathbf{Q}\boldsymbol{A}^p = \tau_p \boldsymbol{A}^p.$$

- τ_p are the delay times and represent the time the energy of the mode described by $\boldsymbol{A}^p$ spends inside the cavity.
- Resonances appear as peaks in the delay spectrum (τ_p vs. k).

Resonances ⟷ Time Delay



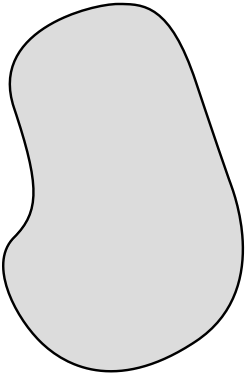
Section Outline

1. Scattering Matrix
   - Resonances / Quasi-Bound States
   - Energy Formalism
   - Time Delay
2. Numerical Construction
3. Applications
   - Numerical Results
   - Outlooks
4. Conclusion

Construction of the $\boldsymbol{S}$-matrix

Goal

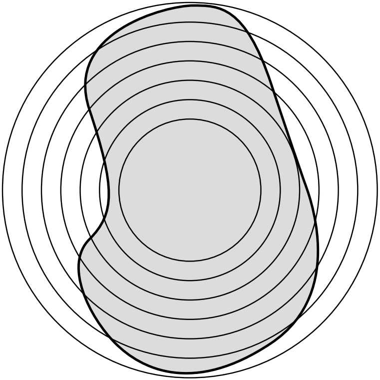
Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



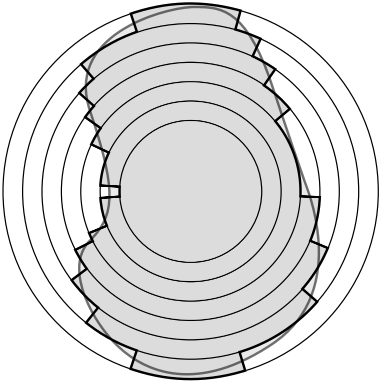
- Divide in concentric shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



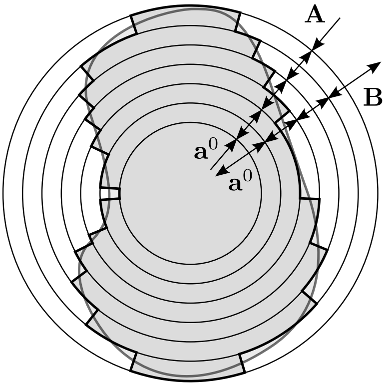
- Divide in concentric shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



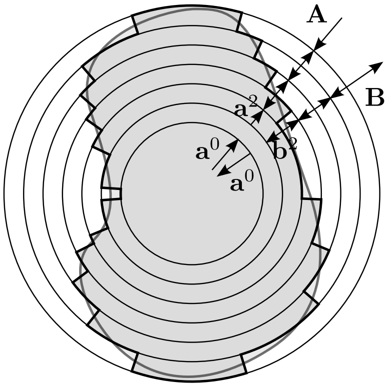
- Divide in concentric shells
- Solve angular part in Fourier series

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



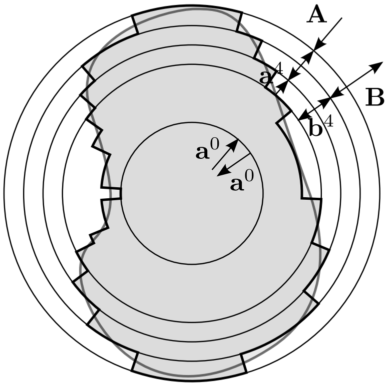
- Divide in concentric shells
- Solve angular part in Fourier series
- Connect the shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



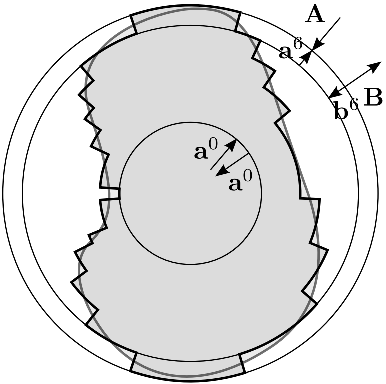
- Divide in concentric shells
- Solve angular part in Fourier series
- Connect the shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



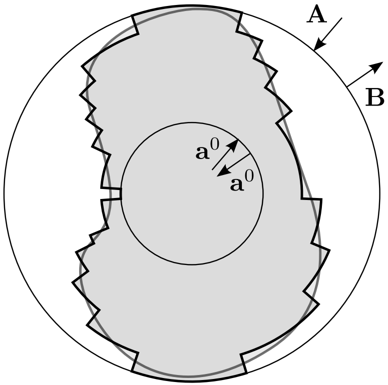
- Divide in concentric shells
- Solve angular part in Fourier series
- Connect the shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



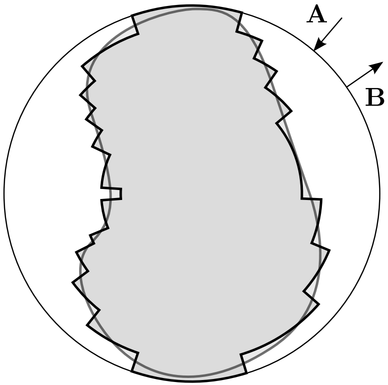
- Divide in concentric shells
- Solve angular part in Fourier series
- Connect the shells

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

Construction of the $\boldsymbol{S}$-matrix

Goal

Compute $\boldsymbol{S}$ for arbitrary geometries and arbitrary refractive index profiles.



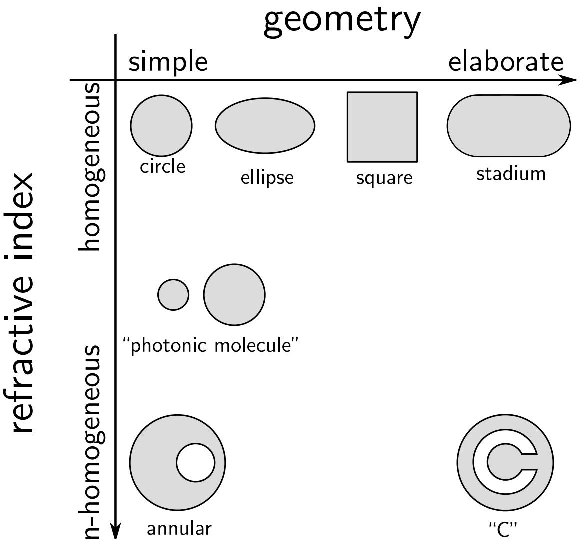
- Divide in concentric shells
- Solve angular part in Fourier series
- Connect the shells
- Isolate $\boldsymbol{S}$ from $\boldsymbol{B} = \boldsymbol{S}\boldsymbol{A}$.

A. I. Rahachou and I. V. Zozoulenko, Appl. Opt. (43), 1761 (2004).

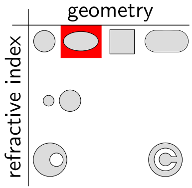
Section Outline

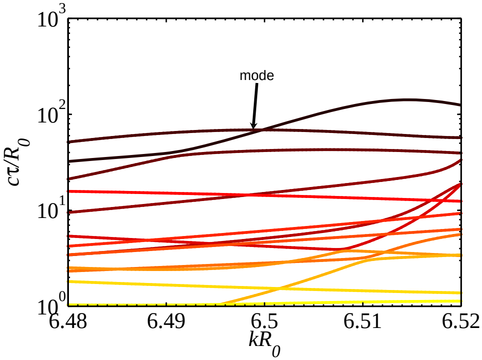
Section Outline

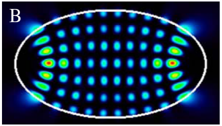
Example Applications of the Method



Ellipse (integrable geometry)



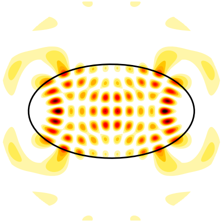




J. Unterhinnighofen *et al*, Phys. Rev. E (**78**), 016201 (2008).

$$\mathrm{Re}\left\{kR_0\right\} = 6.5$$

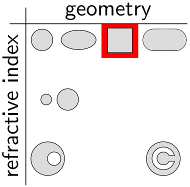
$$|\,\mathrm{Im}\left\{kR_0\right\}| = 0.032$$

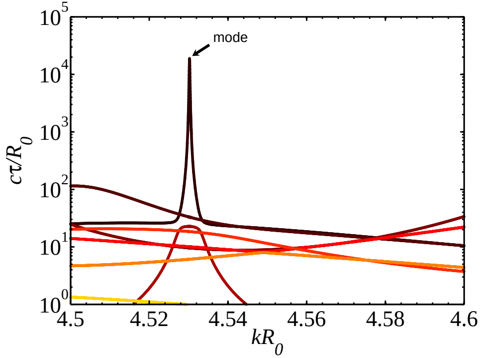


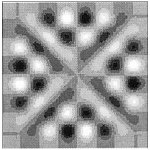
$$kR_0 = 6.499$$

$$2R_0/c\tau = 0.029$$

Square (corners + high delay)



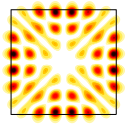




W.-H. Guo *et al*, IEEE J. of Qu. El. (**39**), 1106 (2003).

$$\mathrm{Re}\left\{kR_0\right\} = 4.54$$

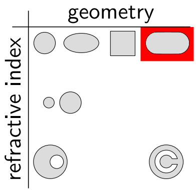
$$|\,\mathrm{Im}\left\{kR_0\right\}| = 1.05 \times 10^{-4}$$

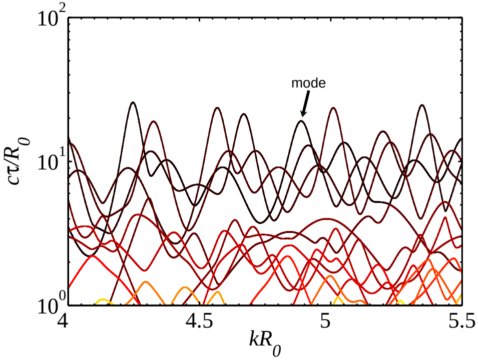


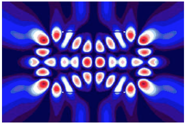
$$kR_0 = 4.53$$

$$2R_0/c\tau = 1.06 \times 10^{-4}$$

Stadium (composite structure)



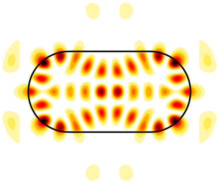




S.-Y. Lee *et al*, Phys. Rev. A (**70**), 023809 (2004).

$$\mathrm{Re}\left\{kR_0\right\} = 4.89$$

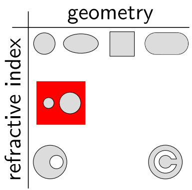
$$\left|\mathrm{Im}\left\{kR_0\right\}\right| = 0.055$$

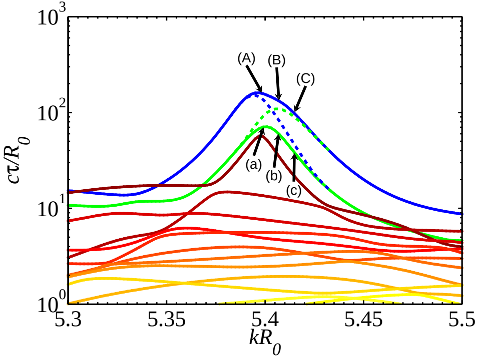


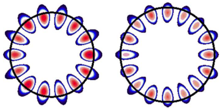
$$kR_0 = 4.89$$

$$2R_0/c\tau = 0.052$$

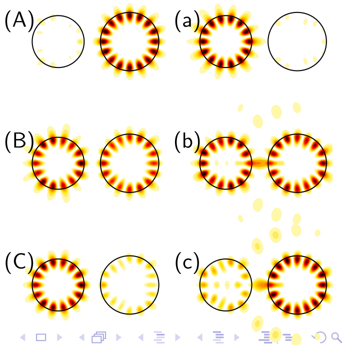
Photonic Complex



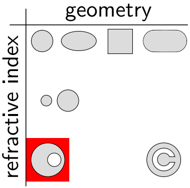


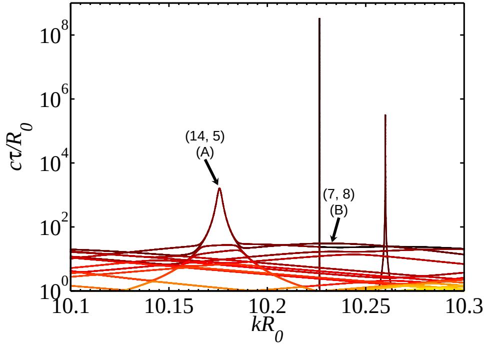


J.-W. Ryu *et al*, Phys. Rev. A (**79**), 053858 (2009).



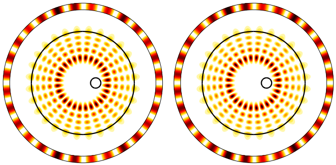
Annular cavity (closed-form, holey cavity)



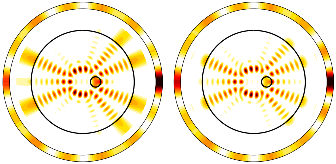


Emission | Scattering

(A) $\mathrm{Re}\{kR_0\} = 10.17570$, $|\mathrm{Im}\{kR_0\}| = 1.2491 \times 10^{-3}$ | $kR_0 = 10.17568$, $2R_0/c\tau = 1.2492 \times 10^{-3}$



(B) $\mathrm{Re}\{kR_0\} = 10.268$, $|\mathrm{Im}\{kR_0\}| = 8.1 \times 10^{-2}$ | $kR_0 = 10.266$, $2R_0/c\tau = 8.3 \times 10^{-2}$



Section Outline

Outlooks

- TE Modes & $\mu \neq 1$
  Equations to solve inside the cavity become

$$\left[\nabla^2 + n^2k^2\right] E_z = \frac{1}{\mu}\nabla\mu \cdot \nabla E_z$$

$$\left[\nabla^2 + n^2k^2\right] H_z = \frac{1}{\epsilon}\nabla\epsilon \cdot \nabla H_z$$

  for TM and TE, respectively, and boundary conditions depend on both μ and ϵ.

- Full 3D
  Possible and theoretically similar, but poses some numerical challenges.
- Steady-state ab initio laser theory (SALT)
  Quasibound states as guides to constant flux states.

Section Outline

Conclusion

Primary Findings

- Scattering description for the $\mathcal{S}$-matrix and its associated delay matrix $\mathbf{Q}$

Conclusion

Primary Findings

- Scattering description for the $\boldsymbol{S}$-matrix and its associated delay matrix $\mathbf{Q}$
- Characteristic modes = eigenvectors $\boldsymbol{A}^p$ of the $\mathbf{Q}$-matrix that undergo a simple phase shift in the presence of the cavity, i.e. self-replicating waves.

Conclusion

Primary Findings

- Scattering description for the $\boldsymbol{S}$-matrix and its associated delay matrix $\mathbf{Q}$
- Characteristic modes = eigenvectors $\boldsymbol{A}^p$ of the $\mathbf{Q}$-matrix that undergo a simple phase shift in the presence of the cavity, i.e. self-replicating waves.
- Resonances = peaks in τ_p vs. k (form a subset of the characteristic modes).

Conclusion

Primary Findings

- Scattering description for the $\boldsymbol{S}$-matrix and its associated delay matrix $\mathbf{Q}$
- Characteristic modes = eigenvectors $\boldsymbol{A}^p$ of the $\mathbf{Q}$-matrix that undergo a simple phase shift in the presence of the cavity, i.e. self-replicating waves.
- Resonances = peaks in τ_p vs. k (form a subset of the characteristic modes).
- Computable quantities: Field in all space, resonance position (real wavenumber), resonance width (real delay) and quality factor Q.

Conclusion

Primary Findings

- Scattering description for the $\boldsymbol{S}$-matrix and its associated delay matrix $\mathbf{Q}$
- Characteristic modes = eigenvectors $\boldsymbol{A}^p$ of the $\mathbf{Q}$-matrix that undergo a simple phase shift in the presence of the cavity, i.e. self-replicating waves.
- Resonances = peaks in τ_p vs. k (form a subset of the characteristic modes).
- Computable quantities: Field in all space, resonance position (real wavenumber), resonance width (real delay) and quality factor Q.
- Approach applicable to open, continuously inhomogeneous cavities of arbitrary geometry.

Acknowledgements

- The ICTON 2013 organizing committee
- The Canada Excellence Research Chair in Photonic Innovations (CERCP), especially Younès Messaddeq and Jean-François Viens.
- The Natural Sciences and Engineering Research Council of Canada (NSERC)
- Computations were made on the supercomputer *Colosse* from Université Laval, managed by Calcul Québec and Compute Canada

NSERC CRSNG

UNIVERSITÉ LAVAL

Calcul Québec

Section Outline

5 Convergence

Homogeneous Disk

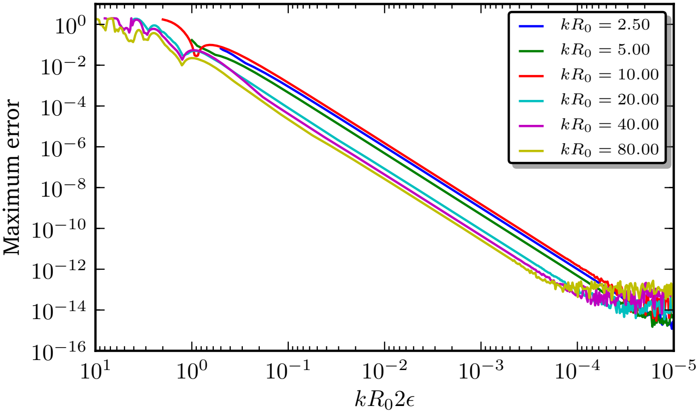


Figure 1 : Maximum error in the elements of the numerical scattering matrix with respect to the size of the annuli.